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Distributed Optimization Approaches with Discrete Variables in the Power Distribution Systems

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***Abstract*—Traditionally, centralized approaches have predominantly been used for the power system operation and control. With increasing penetration of small-scale distributed energy resources (DERs) in the distribution network, especially independently owned renewable resources, distributed algorithms can serve as a potential alternative for improving scalability, resiliency and addressing privacy concerns. However, the complexity of distributed algorithms significantly increases with the integration of the legacy devices, the operation of which depend on discrete control variables. This paper aims to provide a review of the distributed optimization algorithms incorporating discrete control variables for the power distribution system. While the research in this domain is still at its nascence, an extensive comparison of the approaches in the literature for applying quadratic penalty, branch and bound,ordinal optimization and proximal operator to handle discrete variables in the framework of ADMM and dual decomposition have been addressed. Future research direction in this field have been also provided.**

***Index Terms*—Distributed Optimization, Discrete Optimization, Distribution Power System, DERs.**

I. Introduction

WITH increasing penetration of Distributed Energy Resources (DERs), scale of decision variables in power system operation have significantly increased. With expanding high-performance sensor deployment, data acquisition, and processing technology, sharing measurements to a single location raises scalability, security and privacy concerns and increased cyber vulnerabilities. To circumvent these challenges and enhance cyber resiliency, distributed approaches exchange limited information to only a subset of other agents. Furthermore, distributed techniques facilitate reorganizing capabilities in the advent of natural disasters or cyber-attacks, providing much-needed self-healing capabilities.

Needed efficiency for the power distribution system operation has led to the development of several applications in the advanced distribution management system (ADMS). Therefore, notably, many of the applications (e.g., volt-var optimization (VVO), optimal outage management (OMS), distribution service restoration (DSR) of the PDS) deployed within the ADMS utilize various extensions of the Optimal Power Flow (OPF) problem. In addition to the inverter-based resources, these OPF problems need to account for conventional On-load Tap changer (OLTC), switching position of capacitor banks (CB), voltage regulators (VR), reclosers, sectionalizers, and various other switches, the operations of which are not in continuous domain. The requisite non-linear power flow equations, representing the underlying physics of the power system along with discrete control variables lead to the OPF problem becoming NP-hard [1] solving which, even in a centralized setting, can be extremely difficult.

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A multitude of solution frameworks is available in the literature for solving these OPF problems, especially for the PDS. Due to the challenges of solving the NP-hard problem, several relaxations in the power flow equations have been introduced [2]. The authors of [3] propose a mixed-integer quadratic programming deterministic framework to optimally control OLTC, CB, and VRs for VVO. The problem utilizes advanced branch and cut techniques and linearizes power flow equations by considering load to be voltage-dependent as in [4]. In [1], Mixed Integer Second Order Cone Programming (MISOCP) has been implemented in order to combine reactive power optimization with network reconfiguration to minimize power losses and maintain voltage profile in three-phase PDS. In [5], discreteness of VR operations is modeled through an SDP relaxed Branch Flow Model [6], with a generalized Benders Decomposition. Furthermore, [7] proposes a modified Interior point method (IPM) including quadratic penalty terms for non-integer values for discrete decision variables. Notably, all these approaches rely on a centralized optimizer for decision-making.

Typically, distributed algorithms rely on the decomposability of centralized problem formulation or the model itself. The radiality of the PDS helps in achieving the same without a significant increment in computational burden. In this regard, a fully distributed feedback based Volt-Var control algorithm is developed in [8] utilizing augmented Lagrangian Multiplier theory and primal-dual gradients. In [9], the service restoration problem has been solved via ADMM considering DERs and micro-turbines in the PDS. A multi-agent scheme for controlling the DERs to provide voltage support is proposed in [10]. However, these approaches remain silent on optimality guarantee with the introduction of discrete variables.

Solving an OPF with MINLP while guaranteeing optimality can be extremely challenging, especially when all the data is not available at a central location for decision-making. Distributed approaches involving discrete variables are relatively newer, especially in the power system domain. Therefore, to reap the benefits of the distributed methods, it is imperative to analyze the state-of-the-art, especially in the PDS literature. There have been reviews [11], [12] inspecting the existing decentralized/distributed approaches in power system optimization, but these provide no commentary on the convergence of the algorithms in the presence of discrete variables.

This paper aims to bridge the research gap in distribution optimization with discrete variables by critically reviewing the key literature utilizing mathematical programming techniques that account for discrete control variables in the PDS. Quadratic penalty, branch and bound, ordinal optimization and proximal operator method to handle discrete variables in ADMM and dual decomposition have been discussed. We observe that framework encompassing all different kinds of discrete variables are missing in existing literature. Another challenge will be fine-tuning these algorithms in improving the overall efficiency of the optimization algorithm. Comparative analysis of these literature helps in identifying future research directions for distributed OPF with discrete variables.

II. Distributed Optimization and OPF in the PDS

A. Generic ACOPF Formulation

A typical AC optimal Power Flow (ACOPF) problem for the PDS can have multiple forms such as minimization of system losses, voltage or frequency deviation, active power curtailment cost, conservation voltage reduction or service restoration. A generalized constrained optimization problem is shown in (1), where, the objective is to minimize cost $f(x)$ over decision variable set x, satisfying the set of equality and inequality constraints $g(x)$ and $h(x)$ respectively:

$$\min_{x} f(x), \quad s.t., \quad g(x) = 0, \quad h(x) \leqslant 0 \tag{1}$$

A typical Volt-Var optimization problem as a representative ACOPF minimizing the weighted voltage deviations along the distribution feeder has been represented in (2). Typical constraints include nonlinear power flow equations (2c) and power system operational constraints (2b).

$$\min \sum_{i=1}^{N} w_i(|V_i^2| - |V_{ref}^2|)^2 \quad s.t., \tag{2a}$$

$$\underline{V_i} \leq V_i \leq \overline{V_i}; \quad |S_{ij}| \leq \overline{S_{ij}}; \quad -\pi \leq \angle(V_i, V_j^*) \leq \pi \tag{2b}$$

$$\sum_{(i,j)\in\tau^D} Y_{i,j}^* V_i V_i^* = S_i^g - S_i^d \tag{2c}$$

Here, V_i, V_{ref}, S_i^g, S_i^d, S_{ij}, $Y_{i,j}$ identify, network level voltages, voltage references, apparent power generation, apparent power demand, line flows and line admittance respectively.

The non-convexity imposed by nonlinear AC power flow equations necessitates implementation of suitable approximations or relaxations. The main types of relaxations that convexify the ACOPF problem can be categorized to: 1) Semi-Definite Programming, 2) Second-Order Cone Programming and 3) The Linear Relaxation [2]. Furthermore, with the introduction of legacy devices with discrete control steps (3),(4), the problem becomes a mixed integer type problem.

$$\sum_{(i,j)\in\tau^D} S_{ij} = S_i^g - S_i^d - s_i Y_i^C V_i V_i^* \tag{3}$$

$$S_{ij} = \left(Y_{i,j}^s + Y_{i,j}\right)^* \frac{V_i V_i^*}{T_{ij} T_{ij}^*} - Y_{i,j}^* \frac{V_i V_i^*}{T_{ij}} \tag{4}$$

Here, s_i, Y_i^C identify, operational status of capacitor banks, capacitive admittance, respectively. $Y_{i,j}^s$, T_{ij} represent, admittance of tap changing transformers, and complex tap positions, the operation of which are in discrete steps.

To solve the OPF problem, a wide variation of algorithms exist for centrally handling continuous and discrete variables. We observe that, distributed approaches evolve mostly from modification of centralized algorithms with suitable decomposition and coordination. In the following section we describe some of the algorithms that would facilitate detailed discussion in the next section:

B. Approaches for Centralized OPF

1) Interior point Method (IPM): IPM introduces non-negative slack variables to convert the inequality constraints in (1) to $h_t(x) + \mathcal{S}_t = 0$. The sum of the logarithms of all the slack variables forms a 'log barrier' and is added to the objective function with a weighting parameter μ to ensure the non-negativity of slack variables. The objective function in (1) is then reformulated as (5) followed by the corresponding KKT conditions(6).In each iteration, The KKT conditions are solved via Newton's Method and the Lagrange multipliers λ_g *and* λ_h, the barrier parameter μ, variables x and $\mathcal{S}$ are updated accordingly until convergence is achieved.

$$\min f(x) - \mu \sum_{t=1}^{\#ineq} \log \mathcal{S}_t \tag{5}$$

$$\frac{\partial f(x)}{\partial x} + \lambda_h^T \left(\frac{\partial h(x)}{\partial x}\right) + \lambda_g^T \left(\frac{\partial g(x)}{\partial x}\right) = 0 \tag{6a}$$

$$g(x) = 0 \tag{6b}$$

$$h_t(x) + \mathcal{S}_t = 0 \tag{6c}$$

$$diag(\mathcal{S})\gamma - \mu I_{\#ineq} = 0 \tag{6d}$$

2) Sequential Quadratic Programming (SQP): SQP algorithms construct quadratic functions from the generic OPF model (1) and computes search direction $d^{(k)}$ for a given iteration, k, by solving the following equations with modified constraints as (8)

$$d^{(k)} = \underset{d}{argmin} f(x^{(k)}) + \left(\nabla f(x^{(k)})\right)^T d + \frac{1}{2} d^T \left(\nabla_{xx}^2 \mathscr{L}(x^{(k)}, \lambda_h^{(k)}, \lambda_g^{(k)})\right) d \tag{7}$$

$$g_u\left(x^{(k)}\right) + \left(\nabla g_u(x^{(k)})\right)^T d = 0 \quad u = 1, ..., m_{eq} \tag{8a}$$

$$h_t\left(x^{(k)}\right) + \left(\nabla h_t(x^{(k)})\right)^T d \leq 0 \quad t = 1, ..., m_{ineq} \tag{8b}$$

where $\mathscr{L}$ is the Lagrangian with multipliers λ_g and λ_h; ∇ and ∇_{xx}^2 denote the gradient and Hessian, respectively, with respect to x. In each iteration the primal variables are updated as $x(k+1) = x(k) + d(k)$.When the Hessian is difficult to compute, SQP algorithms often employ so called "quasi-Newton" techniques [13].

3) Branch and Bound: An useful strategy to solve combinatorial optimization problem is the Branch and Bound Method (BBM) [14], which is utilized alongside other algorithms for solving mixed integer problems. The BBM iteratively builds a search tree of sub-problems by generating children nodes through partitioning the solution space (branching). It stores a feasible solution globally and updates it whenever the algorithm finds a better solution from evaluating other sub-problems from the set of unexplored problems in further iterations. Regions yielding to suboptimal solutions are pruned from the search space (bounding). Once all sub-problems have been explored the best solution is returned which is proven to be the optimal solution.

C. Approaches for Distributed OPF

In constrained optimization, the proposed distributed techniques in literature can be classified into primal [15], dual [16], or primal-dual [17] based relying on the methodologies used to account for consistency constraints. Here we will review a few of the relevant distributed optimization techniques:

1) Dual Decomposition: If the objective function is separable, the problem in (1) can be reformulated as:

$$\underset{x}{min} \sum_{i=1}^{N} f_i(x_i), \quad s.t. \sum_{i=1}^{N} \mathbf{A}_i x_i = b \tag{9}$$

$$x_i^{k+1} := \underset{x_i}{argmin} \; L_i(x_i, \lambda^k) \tag{10a}$$

$$\lambda^{k+1} := \lambda^k + \alpha^k \left(\sum_{i=1}^{N} (\mathbf{A}_i x_i^{k+1}) - b\right) \tag{10b}$$

This equation can be decomposed in such a way that, each of the controllers solves its Lagrangian $L_i(x_i, \lambda) := f_i(x_i) + \lambda^T \left(\mathbf{A}_i x_i - (1/N)b\right)$ and the primal and dual variables will be iteratively updated as (10).

2) Alternating Direction Method of Multipliers (ADMM): ADMM is one of the widespread distributed approaches which decomposes the problem through dual descent and incorporates method of multipliers for better convergence guarantee [18]. This dual based algorithm reformulates the optimization problem as follows:

$$\min f(x) + g(z), \quad s.t., \; Ax + Bz = c \tag{11}$$

The augmented Lagrangian is formed by including the quadratic penalty term corresponding to the equality constraint for improving the convergence properties,

$$\begin{aligned} \mathscr{L}_\rho(x,z,\lambda) &= f(x) + g(z) + \\ &\lambda^T(Ax + Bz - c) + (\rho/2)\left\|Ax + Bz - c\right\|_2^2 \end{aligned} \tag{12}$$

The x and z minimization steps are conducted by separate agents whereas dual variables are updated centrally as,

$$x^{k+1} := \arg\min \mathscr{L}_\rho(x, z^k, \lambda^k) \tag{13a}$$

$$z^{k+1} := \arg\min \mathscr{L}_\rho(x^{k+1}, z, \lambda^k) \tag{13b}$$

$$\lambda^{k+1} := \lambda^k + \rho(Ax^{k+1} + Bz^{k+1} - c) \tag{13c}$$

One approach for avoiding the central controller and making the algorithm completely distributed is introducing global variables (s_j) associated with the local variables (x_j) and create a bipartite graph between neighboring agents where each edge represents a consensus constraint ($x_j = s_j$). So neighboring agents share information with each other to mutually converge the local variables towards the global variables. The augmented Lagrange function is built as (14) having y as the Lagrange multiplier associated with consensus constraints and N_B^a indicates all the boundary variables of agent a.

$$\begin{aligned} \mathscr{L}_\rho(x,s,y) &= \sum_{a\in A}[f_a(x^a) + \\ \sum_{j\in N_B^a} y_{a,j}^T(x_j^a - s_j) &+ (\rho/2)\sum_{j\in N_B^a}\left\|x_j^a - s_j\right\|^2] \end{aligned} \tag{14}$$

3) Analytical Target Cascading (ATC): ATC method decomposes the Optimization problem with large number of variables into hierarchies of subsystems and each subsystem co-ordinates with its hierarchy elements only for achieving the global convergence. Each element in the hierarchy sets target for its children subsystems in the lower level in preparation of obtaining targets passed by parent subsystem in the upper level. Each controller M_{ij} at level i and subsystem j solves the following problem,

$$\begin{aligned} &\min_{\bar{x}_{ij}}\big[f_{ij}(\bar{x}_{ij}) + \Pi(t_{ij} - r_{ij})\sum_{n=1}^{c_{ij}}\Pi(t_{(i+1)k_n} - r_{(i+1)k_n})\big] \\ &\quad s.t. \;\; g_{ij}(\bar{x}_{ij}) = 0; \quad h_{ij}(\bar{x}_{ij}) \le 0 \\ &\quad where \;\; \bar{x}_{ij} = \left[x_{ij}, t_{(i+1)k_1}, ..., t_{(i+1)k_{c_{ij}}}\right] \in \mathrm{R}^q; \\ &\quad and \;\; r_{ij} = a_{ij}(\bar{x}_{ij}) \end{aligned} \tag{15}$$

here, all local variables to element j at level i is stored in x_{ij}, For any common variable between an element and its parent, a target (t_{ij})-response (r_{ij}) pair is created. Target is decided by the parent and response is calculated from that element's analysis function a_{ij} as in $r_{ij} = a_{ij}(x_{ij}, t_{(i+1)k1}, ..., t_{(i+1)}kc_{ij})$. The deviation function is added to the objective function to match the target and response of each element until convergence.

The strategies for integrating aforementioned distributed approaches with traditional optimization solvers with special modification for handling discrete variables as reported in the literature will be discussed in the next section in detail.

III. Distributed Approaches with Discrete Variables

In each of these approaches, the power system is divided into multiple areas with own controllers. Each controllers coordinate only with controllers of neighbouring areas.

A. ADMM with Extended Interior Point Method

The developed distributed algorithm stems from its centralized counterpart [7] which extends the IPM by adding quadratic terms to the objective function that penalizes the function when the discrete decision variables get other values rather than its nearest rounded off integers. The addition of the penalty term needs to be harmonized so to avoid unnecessary fluctuation in primal and dual residuals. The penalty function for all discrete variables are added to the augmented Lagrangian as (16) where γ_j is the penalty factor corresponding to discrete variable x_j and x_{jb} is determined by rounding the computed value off to its nearest integer. Notably, before such an addition, the discrete variables were relaxed as continuous.

$$\mathscr{L}' = \mathscr{L} + \sum_{j=1}^{p}\frac{1}{2}\gamma_j(x_j - x_{jb})^2 \tag{16}$$

The authors of [19] broadens the aforementioned technique to a distributed algorithm by utilizing ADMM. Auxiliary variables corresponding to boundary node voltages at two ends of a tie line are introduced for ADMM consensus and the optimization problem is reformulated as in (14). The requisite algorithm is provided in Algo. 1:

Algorithm 1: ADMM with EIPM

Data: Voltage & Power Flow Measurements from PDS
Result: Optimum Solution for DER Generation and tap position of OLTC, CB

```
Set all auxiliary variables to unity;
do            /* Outer loop iterations */
    Solve local optimization problems through EIPM;
    Add integer penalty terms when needed;
    do        /* Inner loop iterations */
        Obtain solutions for primal and auxiliary
          variables through local computation from
          local and shared variables and consensus
          constraints ;
        Send boundary variables to neighbors;
        Update dual variables corresponding to
          equality, inequality and consensus constraints;
    while Repeated solutions of ADMM iterations
      converge?[Check];
    Update primal and dual variables;
    Calculate primal and dual residuals;
while Repeated solutions of local EIPM method
  converge?[Check];
```

B. Incremental ADMM with Extended Interior Point Method

Alternatively, the optimization problem of Approach A is reshaped in [19] by perturbing the KKT conditions for incremental values of variables instead. Linearization gives rise to following set of equations,

$$\min_{\Delta x, \Delta s}\sum_{a\in A}\left(\frac{1}{2}(\Delta x^a)^T H^a \Delta x^a - (\mathscr{L}_{x^a}^a)^T\Delta x^a)\right) \tag{17a}$$

$$s.t. \quad J^a\Delta x^a = \mathscr{L}_{\lambda_a}^a, a\in A, (\Delta\lambda) \tag{17b}$$

$$\Delta x_j^a - \Delta s_j = 0, \vee j \in N_B^a, \vee a \in A, (\Delta y) \tag{17c}$$

Here, J^a is the Jacobian matrix corresponding to the equality constraints, H^a denotes the Hessian matrix, $\mathscr{L}_\lambda$ and $\mathscr{L}_x$ are the residual vectors of KKT conditions. It can be observed that, the optimization sub problem is simplified to a quadratic problem (QP) in terms of local variable increments which only needs incremental boundary variables to be shared among neighbors. Thus the problem becomes an Incremental Oriented (IO) ADMM. The problem is solved in the same manner as previous IPM. The results in [19] show that, the IO-ADMM improves the convergence and quality of solution significantly.

C. ADMM with Branch and Bound

The method in [20] utilizes ADMM in conjunction with BBM to account for the discrete tap positions of OLTC in a distributed manner. Since ADMM convergence is only guaranteed for convex problems [18], angle and SOCP relaxations are utilized. Initially, discrete variables are linearly relaxed and global variables are introduced for the boundary variables to converge through consensus constraints (14). Once the ADMM converges, BBM is applied to the discrete variables giving rise to 2^n candidate solutions, each associated with the tap changer position; which is solved using B&B method, as discussed in Algo. 2.

Algorithm 2: ADMM with Branch and Bound

Data: Solution with continuous relaxation
Result: Optimum Solution for tap position of OLTC
Initialize objective function to ∞;
while *unretrieved branch* $\neq \varnothing$ **do**
  Initialize boundary variables as measured data;
  Set Lagrange Multipliers to zero;
  do `/* ADMM iterations */`
    Solve the minimization problem considering branch constraints;
    if *Solution not feasible?* **then**
      Fetch nearest unretrieved branch, and start over;
    end
    Update global variables, Lagrangian multipliers, residuals and penalty parameters;
    Exchange boundary variables and residuals to adjacent regions;
  while *Residuals Converge?*;
  Exchange boundary voltages and objective functions ;
  if *Cost improved?* **then**
    Add a cut to upper limit of the objective ;
  end
  Calculate and exchange tap positions among adjacent regions;
  if *Tap positions are integers?* **then**
    Reconstruct branches considering voltage related constraints;
  else
end

D. Dual Decomposition with SQP implementing DPQN and Ordinal Optimization (OO)

Algorithm presented in [21] decomposes the entire optimization problem into subsystems each solving its local minimization problem by employing dual pseudo quasi Newton (DPQN) method as part of SQP relaxing all the discrete variables as continuous. The generic formulation of (7) is suitably decomposed (18) where Δx is the increment in decision variable to be added in the next iteration,

$$\min_{\Delta x_i \in \Omega_i} [\Delta x_i^T \nabla^2_{x_i} f_i \Delta x_i + \nabla_{x_i} f_i^T \Delta x_i] + \overset{o}{\lambda}{}_i^T [\overset{o}{g}_i + \nabla_{x_i} \overset{o}{g}{}_i^T \Delta x_i] + \lambda_{ib}^T [g_{ib} + \nabla_{x_i} g_{ib}^T \Delta x_i] + \sum_{j \in N_i} [\lambda_{jb}^{iT} \nabla_{x_{ib}^j} g_{jb}^{iT} \Delta x_{ib}^j] \quad (18)$$

Here, g_{ib}, λ_{ib} and $\overset{o}{g}_i, \overset{o}{\lambda}_i$ represent the equality constraints pertaining to boundary buses and local buses respectively. Thus the original problem is decomposed and lagrange multipliers from only the neighboring buses (N_i) are needed by any local controller. After the distributed solution with continuous variables are found, OO strategy is implemented via a root subsystem for selecting a sufficiently reasonable combination. Coordination by root-system can be akin to centralized algorithm. The surrogate model here is built based on the sensitivity theory described in [22]. Overall algorithm has been summarized in Algo. 3:

Algorithm 3: SQP with DPQN and OO

Data: Voltage & Power Flow measurements from PDS
Result: Optimum Solution for DER Generation and tap position of OLTC, CB
Command from Root Subsystem: Solve the SQP with DPQN method considering continuous relaxation of the discrete variables:: **Problem-Distributed**;
Stage 1: Set variables which are already near a integer value based on pre-defined threshold ;
Stage 2: Perform the sensitivity analysis as per [22] to rank potential candidate solution set is separated ;
do
  Stage 3: Solve **Problem-Distributed** based on outcome of **Stage 2** and determined initial solution in **Stage 1** ;
while *Good Enough Solution Found?*;

E. ADMM with Relax-Drive-Polish Algorithm

The work on [23] solves the DSR problem after an power outage through network reconfiguration and load restoration by proposing a heuristic relax-drive-polish algorithm in conjunction with ADMM modified from [24]. PDS constraints are linearized, and therefore, overall problem is mixed integer convex-programming (MICP). For DSR, the discrete variables are energization states of tie switches, buses and load pickup. Another set of discrete variables arise from the topological constraints to ensure radiality of the system. In this paper, binding variables correspond to constraints that prevent the clusters to decompose the problem in a complete distributed way whereas other variables are called inner variables to each cluster. The Algorithm can be summarized in Algo. 4.

Algorithm 4: ADMM with Relax-Drive-Polish

Data: Voltage & Power Flow measurements from PDS
Result: Near Optimum Solution for Tie Switch status, Bus status and Load pickup
Relax all binary variables; Start ADMM iterations. Find a preliminary point for warm-start;
Drive the solution of binary variables to boolean values by incorporating proximal operator and regularization parameter instead of direct projection for better convergence;
Fix the binding binary variables and finalize the network configuration;
Polish the existing solution by another set of ADMM iterations through a Mixed integer Solver. In this stage, the bus energization states and load pickups are finalized;

F. ADMM with Projection Method

Strategy proposed in [25] is for DSR through network reconfiguration and distributed generators set-point allocation through combining ADMM with projection method. The network is divided into clusters based on location of tie switches, each cluster to be controlled by a local controller which decides the operating status of the tie switches not violating radiality constraints [26]. So, the discrete variables are energization states of tie switches, and variables ensuring radiality. The Algorithm can be summarized as follows:

Algorithm 5: ADMM with Projection Method

Data: Voltage & Power Flow measurements from PDS
Result: Near Optimum Solution for Tie Switch status and Load pickup
Relax the global binary variables related to radiality constraints;
Partition primal variables to two components, one containing only continuous variables, another containing mixed-integer;
do
  Solve Sub-problem I through ADMM iterations for continuous variables;
  Solve relaxed sub-problem II considering solution of Solve Sub-problem I and projecting the binary variables to nearest boolean value;
  Update Lagrange multipliers and share with neighbors;
while *Convergence?*;

IV. Discussions

We further compare and contrast the discussed approaches corresponding to distributed algorithms involving discrete variables in the PDS in Table I. Relaxation of the discrete variables, and relaxation of AC power flow equation and subsequent rounding seems to be mostly common. In this table we discuss typical model used for determining the control variables, the objective function considered, methods to solve problem associated with discrete variables with distributed optimization, necessary exchange of boundary variables, requisite communication, decision variable considered as a part of optimization problem, and observed pros and cons of the algorithms. The comparison table is expected to provide us with an insight of the path forward. We observe that while these algorithms can account for discrete variables, their scope of operation is limited, e.g., a typical algorithm designed to determine optimal tap position may not be suitable for determining optimal switching operation of the capacitor banks. This is dependent upon how the algorithm handles the discrete variables. We observe that some approaches need more boundary information to be shared which may increase the communication burden. Additionally, majority of these approaches are first-order (they uses ADMM/Dual-decomposition method).

Furthermore, all of these aforementioned techniques divide the PDS into a number of areas for designing the area controllers. Besides, we observe the absence of a standardized way of incorporating different types of discrete variables, which significantly limits the applicability of the existing approaches. Notably, the ways to account for binary variables in distributed optimization and control application is not limited to power system domain, and have been an important field of research in various other engineering applications giving rise to developments of diversified strategies. The work in [27] proposes a technique of integrating ATC with BBM to solve the MINLP problem in a hierarchical manner among multiple controllers. BBM works as the outer loop whereas ATC performs the inner loop iterations. The authors of [28] discusses on implementing cutting planes to solve MILP problems in distributed manner. To add, though we have emphasized the scope of this paper within mathematical programming based approaches, development of machine learning based decentralized techniques in power system domain has also gained attention among researchers in recent years [29], [30].

However, as it is well known, the power system has certain unique properties, which could certainly be leveraged for efficient controller development. It would be worth investigating whether a generic formulation encompassing typicality of various controller devices is possible, or each of the controller needs to be custom made. We also need to investigate how we can increase the speed of computation without much sacrifice on the solution quality. Also, future work can be accomplished on making these algorithms robust to cyber and communication uncertainties and failures.

V. Conclusions

In this work, a comprehensive literature review has been conducted to identify existing methodologies to solve optimal power flow problem involving discrete variable for a distributed optimization. We have provided a detailed discussions to understand the underlying algorithm, and subsequently compared all these algorithms across a set of parameters. We observe that even with limited literature availability, there is clearly a lack of available framework that encompasses all different kinds of discrete variables. While we have noted the existences of different kinds of algorithms in various other areas, the challenges would be how these algorithms could be fine-tuned to leverage some of the properties of the power system, improving the overall efficiency of the distributed optimization and control algorithm.

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TABLE I
COMPARISON ON THE SCOPE OF THE REVIEWED APPROACHES

Approach	Problem specification	Objective function (minimizing)	Discrete Algorithm	Distributed Algo.	Boundary Variables	Comm. Requirements	Decision Variables	Comment
A	MINLP	1. Real Power Generation 2. System loss	Quadratic penalty term for non integer values	ADMM	Auxiliary variables representing the real and imaginary part of voltages at the boundary buses	Neighbor agents only	1. OLTC Tap 2. Cap bank switching	– May diverge due to nonconvexity.
B	MIQP	1. Real Power Generation 2. System loss	Quadratic penalty term for non integer incremental values	ADMM	Auxiliary variables representing the increments in real and imaginary part of voltages at boundary buses	Neighbor agents only	1. OLTC Tap 2. Cap bank switching	– Convergence guarantee with ADMM. – Better privacy; only sharing the increments rather than actual values.
C	MISOCP with Cutting Planes and Angle Relaxation	1. Active power curtailment cost, 2. System Loss	Branch and Bound	ADMM	Tie-line P and Q, primal and dual residual, boundary node voltage, objective function value of upstream and downstream region, SVR tap position	Neighbor agents and root subsystem	OLTC Tap	– Optimality and convergence guaranteed. – Capacitor bank position not considered.
D	MIQP	1. Real Power Generation 2. System Loss	Ordinal Optimization	Dual Decomposition	Lagrange Multipliers associated with boundary buses	Neighbor agents and root subsystem	1. OLTC Tap 2. Cap bank switching	– Root Subsystem is needed for discrete variables. – Sub-optimal but good enough solution. – No convergence guarantee.
E	MICP with linearized constraints	1. Unrestored Load After Outage 2. Change in network topology	Proximal Operator with Projection	ADMM	Tie Line P & Q, Boundary Node voltage Lagrange Multipliers associated with boundary buses variables for radiality	Neighbor agents only	1. Tie-switch status 2. Load pickup 3. Bus Status	– Binary Variables inner to cluster are determined by MICP solver – Convergence relies on ADMM parameter
F	MISOCP	1. De-energized Zone 2. Change in network topology 3. Load Shedding 4.Cost of Energy production 5. System Loss	Relaxed and then Projected	ADMM	Tie Line P & Q, Boundary Node voltage Lagrange Multipliers associated with boundary buses variables for radiality	Neighbor agents only	1. Tie-switch status 2. Load pickup	– Direct projection may fluctuate convergence – convergence is not guaranteed

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